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A STUDY ON TORSION OF A NON-LINEAR VISCOELASTIC SLAB ABOUT NON-COINCIDENT AXES

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Abstract—In this work, the torsion of a non-linear viscoelastic slab between two infinite plates about non-coincident axes, where the top and the bottom boundary surfaces are bounded to rigid plates, is studied. The boundary-initial value problem is formulated and is solved numerically, with deformation prescribed at boundaries. The numerical procedure is such that at each time step, the problem is equivalent to a system of three coupled non-linear partial integro-differential equations for time and the displacement functions.

1. INTRODUCTION

In recent years, due to the development of modern polymeric materials, there has been a considerable amount of interest in the study of inhomogeneous deformations in non-linear viscoelastic solids. However, the advent of theoretical study has not kept pace with the advances made in industry. Much of the theoretical work has been limited to modeling the linear response range of the solids. Thus, when attention is focused on truly non-linear viscoelastic models, one is hardpressed to find many boundary-initial value problems which have been solved. Wineman (1972, 1978) studied the response of axially symmetric non-linear viscoelastic membranes. Recently, Dai *et al.* (1992) studied the non-uniform extension of a non-linear viscoelastic slab. Dai and Rajagopal (1993) investigated the proportional shearing of a non-linear viscoelastic layer. They studied two boundary-initial value problems; one in which the slab displacement is specified and the other in which the traction on the boundary is prescribed. They found that inhomogeneous deformations are possible for both types of boundary-initial value problems.

This paper studies the torsion of a non-linear viscoelastic slab about non-coincident axes, which is an extension to non-linear viscoelasticity of the work established by Rajagopal and Wineman (1985) in finite elasticity. The problem under consideration is concerned with the deformation of a viscoelastic slab of thickness $2H$ whose other dimensions are infinite, the top and bottom surfaces of the sandwich being bonded to rigid plates. The aim of this work is to investigate the consequences of shearing and rotating the top and bottom plates by constants about non-coincident axes perpendicular to the plates (before shearing, the axes are coincident, cf. Fig. 1). For the non-linear viscoelastic material between the plates, the constitutive equation is used whose form originally was proposed by Pipkin and Rogers (1968) and was adapted by Wineman (1972, 1978), so that when the time dependence is suppressed in the constitutive expression for the stress, the model reduces to the classical

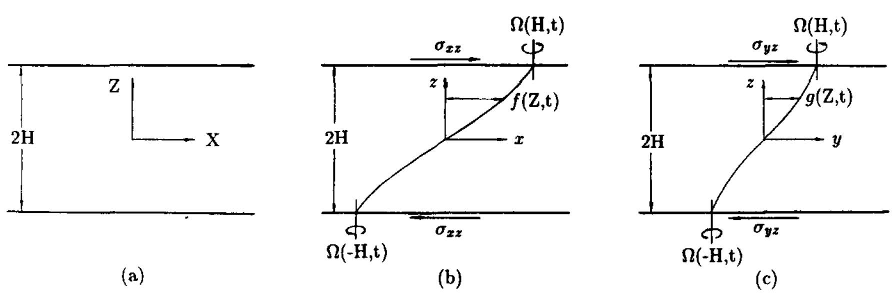

Fig. 1. Domain of deformation.

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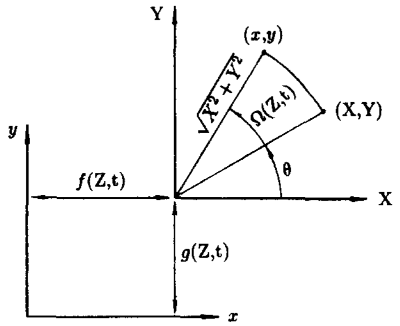

Fig. 2. Displacement of a material point.

Mooney–Rivlin model of finite elasticity. The boundary-initial value problem is studied in which the slab displacement is specified.

In the next section, the kinematics of the problem are discussed and the constitutive equation for the non-linear viscoelastic solid is introduced. The non-coincident rotation problem corresponding to relaxation is formulated in Section 3. The governing equations reduce to a system of three coupled non-linear partial integro-differential equations. A brief description of the numerical procedure to solve the equations is given in Section 4. Section 5 is devoted to discussion of the results.

2. KINEMATICS

For the problem under consideration, it would be appropriate to assume a motion of the form (cf. Rajagopal and Wineman, 1985)

$$x = X\cos\Omega(Z,t) - Y\sin\Omega(Z,t) + f(Z,t),$$

$$y = X\sin\Omega(Z,t) + Y\cos\Omega(Z,t) + g(Z,t),$$

$$z = Z, \tag{1}$$

where X, Y, Z and x, y, z represent the reference and the current coordinates of the same material point. The parameter t denotes the current time. The above motion represents a deformation in which material points that lie in any plane parallel to the boundary plates are translated by $f(Z, t)$ and $g(Z, t)$ in the X- and Y-directions, respectively [cf. Fig. 1(b, c)], and continue to remain in the plane, the plane rotating about the point at $x = f(Z, t)$ and $y = g(Z, t)$ by an amount $\Omega(Z, t)$ (cf. Fig. 2). The locus of these centers of rotation, f and g, is in general a curve in space passing through the centers of rotation of the top and bottom plates.

Then, the deformation gradient $\mathbf{F}$ is given by

$$\mathbf{F} = \begin{pmatrix} C & -S & -XS\Omega' - YC\Omega' + f' \\ S & C & XC\Omega' - YS\Omega' + g' \\ 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

where the prime denotes differentiation with respect to Z and

$$C = \cos\Omega(Z,t), \quad S = \sin\Omega(Z,t).$$

It follows from eqn (2) that the Cauchy–Green strain tensor $\mathbf{B} \equiv \mathbf{F}\mathbf{F}^{\mathrm{T}}$ has the following matrix representation:

$$\mathbf{B} = \begin{pmatrix} 1+\alpha^2 & \alpha\beta & \alpha \\ \alpha\beta & 1+\beta^2 & \beta \\ \alpha & \beta & 1 \end{pmatrix}, \tag{3}$$

where

$$\alpha = -XS\Omega' - YC\Omega' + f', \quad \beta = XC\Omega' - YS\Omega' + g'.$$

Thus, $\mathbf{B}^2 \equiv \mathbf{BB}$ yields

$$\mathbf{B}^2 = \begin{pmatrix} (1+\alpha^2)^2+\alpha^2(1+\beta^2) & \alpha\beta(3+\alpha^2+\beta^2) & \alpha(2+\alpha^2+\beta^2) \\ \alpha\beta(3+\alpha^2+\beta^2) & (1+\beta^2)^2+\beta^2(1+\alpha^2) & \beta(2+\alpha^2+\beta^2) \\ \alpha(2+\alpha^2+\beta^2) & \beta(2+\alpha^2+\beta^2) & 1+\alpha^2+\beta^2 \end{pmatrix}. \tag{4}$$

The deformation tensor $\mathbf{C} \equiv \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is given by

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & C\alpha+S\beta \\ 0 & 1 & C\beta-S\alpha \\ C\alpha+S\beta & C\beta-S\alpha & 1+\alpha^2+\beta^2 \end{pmatrix}. \tag{5}$$

Also, the tensor $\mathbf{F}^{-1}$ has the matrix representation:

$$\mathbf{F}^{-1} = \begin{pmatrix} C & S & -(C\alpha+S\beta) \\ -S & C & S\alpha-C\beta \\ 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

For notational convenience, let $\mathbf{M} \equiv \mathbf{FC}(s)\mathbf{F}^{\mathrm{T}}$, where s is the time in history. From eqns (2) and (5), one observes that the components of $\mathbf{M}$ are:

$$M_{11} = 1+\alpha[2C\phi_1(s)-2S\phi_2(s)+\alpha\phi_3(s)],$$

$$M_{12} = M_{21} = (C\beta+S\alpha)\phi_1(s)+(C\alpha-S\beta)\phi_2(s)+\alpha\beta\phi_3(s),$$

$$M_{13} = M_{31} = C\phi_1(s)-S\phi_2(s)+\alpha\phi_3(s),$$

$$M_{22} = 1+\beta[2S\phi_1(s)+2C\phi_2(s)+\beta\phi_3(s)],$$

$$M_{23} = M_{32} = S\phi_1(s)+C\phi_2(s)+\beta\phi_3(s),$$

$$M_{33} = \phi_3(s), \tag{7}$$

where

$$S(s) = \sin\Omega(Z,s),$$

$$C(s) = \cos\Omega(Z,s),$$

$$\Omega(s) = \Omega(Z,s),$$

$$\Omega'(s) = \Omega'(Z,s),$$

$$\alpha(s) = -XS(s)\Omega'(s)-YC(s)\Omega'(s)+f'(s),$$

$$\beta(s) = XC(s)\Omega'(s) - YS(s)\Omega'(s) + g'(s),$$

$$\phi_1(s) = C(s)\alpha(s) + S(s)\beta(s),$$

$$\phi_2(s) = C(s)\beta(s) - S(s)\alpha(s),$$

$$\phi_3(s) = 1 + \alpha^2(s) + \beta^2(s).$$

In this paper, the same constitutive relation for the Cauchy stress $\boldsymbol{\sigma}$ as that used by Wineman (1973, 1976), Dai *et al.* (1992) and Dai and Rajagopal (1993) is assumed, which is given below

$$\boldsymbol{\sigma} = -p\mathbf{I} + C_0[(1+\mu I)\mathbf{B} - \mu\mathbf{B}^2]$$
$$-\frac{C_0(1-\gamma)}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)\{[1+\mu I(s)]\mathbf{B} - \mu\mathbf{F}\mathbf{C}(s)\mathbf{F}^T\}\,\mathrm{d}s, \tag{8}$$

where $\mathbf{C}(s)$ is obtained through replacing the current time t by the history time s and

$$I = \mathrm{tr}\mathbf{C}, \quad I(s) = \mathrm{tr}\mathbf{C}(s).$$

Note that if time dependence is suppressed from eqn (8), it reduces to a Mooney–Rivlin elastic material.

3. EQUATION OF MOTION

For computational convenience, the equations of equilibrium are expressed in terms of the reference configuration. The quasistatic problem is to be studied and thus t will be considered as a parameter. In the absence of body forces, a lengthy but straightforward calculation using eqns (2)–(8) yields the following equations of motion (wherein the initial terms have been neglected):

$$3C_0\mu\beta\Omega' - C\frac{\partial p}{\partial X} + S\frac{\partial p}{\partial Y} - \frac{C_0(1-\gamma)\mu}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)$$
$$\times[\Omega' + 2\Omega'(s)][S\phi_1(s) + C\phi_2(s)]\,\mathrm{d}s + C_0\frac{\partial}{\partial Z}\left\{(1+\mu)\alpha\right.$$
$$\left.-\frac{1-\gamma}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)\times\{(1+2\mu)\alpha - \mu[C\phi_1(s) - S\phi_2(s)]\}\,\mathrm{d}s\right\} = 0, \tag{9}$$

$$3C_0\mu\alpha\Omega' + S\frac{\partial p}{\partial X} + C\frac{\partial p}{\partial Y} + \frac{C_0(1-\gamma)\mu}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)$$
$$\times[\Omega' + 2\Omega'(s)][S\phi_2(s) - C\phi_1(s)]\,\mathrm{d}s + C_0\frac{\partial}{\partial Z}\left\{(1+\mu)\beta\right.$$
$$\left.-\frac{1-\gamma}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)\{(1+2\mu)\beta - \mu[S\phi_1(s) + C\phi_2(s)]\}\,\mathrm{d}s\right\} = 0, \tag{10}$$

$$(C\alpha + S\beta)\frac{\partial p}{\partial X} + (C\beta - S\alpha)\frac{\partial p}{\partial Y} - \frac{\partial p}{\partial Z} = 0. \tag{11}$$

It can be shown that the following governing equations for the problem under consideration are obtained from the above system:

$$\left\{1+\mu-(1-\gamma)(1+2\mu)\left[1-\exp\left(-\frac{t}{\tau_R}\right)\right]\right\}\Omega''$$

$$+\frac{(1-\gamma)\mu}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)\Omega''(s)\,\mathrm{d}s = 0, \tag{12}$$

$$\left\{1+\mu-(1-\gamma)(1+2\mu)\left[1-\exp\left(-\frac{t}{\tau_R}\right)\right]\right\}(\Omega'^2 f'+f''')+3\mu(-\Omega'^2 f'+\Omega'' g'+\Omega' g'')$$

$$-\frac{(1-\gamma)\mu}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)\Big\{-[S(s)S+C(s)C]f'''(s)+[C(s)S-S(s)C]g'''(s)$$

$$+[3\Omega'+2\Omega''+\Omega''(s)]\{S[C(s)f'(s)+S(s)g'(s)]-C[S(s)f'(s)-C(s)g'(s)]\}$$

$$+[2\Omega'^2-\Omega'(s)\Omega'-\Omega'^2(s)]\{S[S(s)f'(s)-C(s)g'(s)]+C[C(s)f'(s)+S(s)g'(s)]\}$$

$$-2\Omega'(s)f'[\Omega'+\Omega'(s)]\Big\}\,\mathrm{d}s = 0, \tag{13}$$

$$\left\{1+\mu-(1-\gamma)(1+2\mu)\left[1-\exp\left(-\frac{t}{\tau_R}\right)\right]\right\}(\Omega'^2 g'+g''')-3\mu(\Omega'' f'+\Omega' f''+\Omega'^2 g')$$

$$+\frac{(1-\gamma)\mu}{\tau_R}\int_0^t \exp\left(-\frac{t-s}{\tau_R}\right)\Big\{[C(s)S-S(s)C]f'''(s)+[S(s)S+C(s)C]g'''(s)$$

$$+[3\Omega'+2\Omega''+\Omega''(s)]\{S[S(s)f'(s)-C(s)g'(s)]+C[C(s)f'(s)+S(s)g'(s)]\}$$

$$-[2\Omega'^2-\Omega'(s)\Omega'-\Omega'^2(s)]\{S[C(s)f'(s)+S(s)g'(s)]-C[C(s)f'(s)+S(s)g'(s)]\}$$

$$+2\Omega'(s)g'[\Omega'+\Omega'(s)]\Big\}\,\mathrm{d}s = 0. \tag{14}$$

Equations (12)–(14) contain three unknown functions, Ω, f and g, and their derivatives.

Now, attention is turned to considering boundary conditions. From the coupled system of eqns (12)–(14), it is seen that eight boundary conditions are required to make the problem determinate, since eqn (12) is a second order differential equation for Ω, and eqns (13) and (14) are third order differential equations for g and f. A number of solutions can be obtained through rotating the top and bottom plates by differing amounts. In this work the torsion problem of rotating the top and bottom by the same amount in opposite directions will be solved. Suppose the viscoelastic slab is bonded by the planes $Z=-H$ and $Z=H$. Thus, the six appropriate conditions at the top and bottom boundaries are

$$u(0,0,-H,t) = -a(t), \quad u(0,0,H,t) = a(t), \tag{15}$$

$$v(0,0,-H,t) = -b(t), \quad v(0,0,H,t) = b(t), \tag{16}$$

$$\Omega(-H,t) = -\Omega_0(t), \quad \Omega(H,t) = \Omega_0(t), \tag{17}$$

where u and v are the displacements in the X- and Y-directions, respectively, and a, b and

Ω_0 denote the displacements in the X- and Y-directions and the rotation angle on the boundaries, respectively, which are, in general, functions of t. Without losing any generality, it is assumed that $b(t) = 0$. Thus, the displacements on the boundaries prescribed by (15) and (16) can be written as

$$f(-H,t) = -a(t), \quad f(H,t) = a(t), \tag{18}$$

$$g(-H,t) = 0, \quad g(H,t) = 0. \tag{19}$$

Two more boundary conditions are needed to make the problem complete, and these are given below:

$$f'(-H,t) = f'(H,t), \quad g'(-H,t) = g'(H,t). \tag{20}$$

4. NUMERICAL METHOD

The numerical method used to solve eqns (12)–(14), is described in this section. The variables,

$$\frac{\partial\Omega'(Z,t)}{\partial Z}, \quad \frac{\partial f''(Z,t)}{\partial Z} \quad \text{and} \quad \frac{\partial g''(Z,t)}{\partial Z},$$

are solved at each time step in terms of $\Omega(Z,\ t)$, $\Omega'(Z,\ t)$, $f(Z,\ t)$, $f'(Z,\ t)$, $f''(Z,\ t)$, $g(Z,\ t)$, $g'(Z,\ t)$ and $g''(Z,\ t)$, and the solutions at past times. For convenience, eqns (12)–(14) are rewritten as:

$$F_1[t]\frac{\partial\Omega'}{\partial Z} + \int_0^t \frac{\partial\Omega'(s)}{\partial Z} G_1[t-s]\,\mathrm{d}s = 0, \tag{21}$$

$$\begin{aligned} &F_1[t]\frac{\partial f''}{\partial Z} + F_2[\Omega',\Omega'';f';g',g'';t] \\ &+ \int_0^t \frac{\partial f''(s)}{\partial Z} G_2[\Omega;\Omega(s);t-s]\,\mathrm{d}s + \int_0^t \frac{\partial g''(s)}{\partial Z} G_3[\Omega;\Omega(s);t-s]\,\mathrm{d}s \\ &+ \int_0^t G_4[\Omega,\Omega',\Omega'';\Omega(s),\Omega'(s),\Omega''(s);f';f'(s);g'(s);t-s]\,\mathrm{d}s = 0, \end{aligned} \tag{22}$$

$$\begin{aligned} &F_1[t]\frac{\partial g''}{\partial Z} + F_3[\Omega',\Omega'';f',f'';g';t] \\ &+ \int_0^t \frac{\partial f''(s)}{\partial Z} G_3[\Omega;\Omega(s);t-s]\,\mathrm{d}s + \int_0^t \frac{\partial g''(s)}{\partial Z} G_2[\Omega;\Omega(s);t-s]\,\mathrm{d}s \\ &+ \int_0^t G_5[\Omega,\Omega',\Omega'';\Omega(s),\Omega'(s),\Omega''(s);f'(s);g';g'(s);t-s]\,\mathrm{d}s = 0, \end{aligned} \tag{23}$$

where

$$F_1 = 1+\mu-(1-\gamma)(1+2\mu)\left[1-\exp\left(-\frac{t}{\tau_R}\right)\right],$$

$$F_2 = 3\mu(-\Omega'^2 f' + \Omega'' g' + \Omega' g'') + F_1\Omega'^2 f',$$

$$F_3 = -3\mu(\Omega'' f' + \Omega' f'' + \Omega'^2 g') + F_1\Omega'^2 g',$$

$$G_1 = -\frac{\mu(1-\gamma)}{\tau_R}\exp\left(-\frac{t-s}{\tau_R}\right),$$

$$G_2 = [S(s)S+C(s)C]\frac{\mu(1-\gamma)}{\tau_R}\exp\left(-\frac{t-s}{\tau_R}\right),$$

$$G_3 = -[C(s)S-S(s)C]\frac{\mu(1-\gamma)}{\tau_R}\exp\left(-\frac{t-s}{\tau_R}\right),$$

$$G_4 = \Big\{2\Omega'(s)f'[\Omega'+\Omega'(s)]-[3\Omega'+2\Omega''+\Omega''(s)]\{S[C(s)f'(s)+S(s)g'(s)]$$

$$-C[S(s)f'(s)-C(s)g'(s)]\}-[2\Omega'^2-\Omega'(s)\Omega'-\Omega'^2(s)]$$

$$\{S[S(s)f'(s)-C(s)g'(s)]+C[C(s)f'(s)+S(s)g'(s)]\}\Big\}\frac{\mu(1-\gamma)}{\tau_R}\exp\left(-\frac{t-s}{\tau_R}\right),$$

$$G_5 = \Big\{2\Omega'(s)g'[\Omega'+\Omega'(s)]+[3\Omega'+2\Omega''+\Omega''(s)]\{S[S(s)f'(s)-C(s)g'(s)]$$

$$+C[C(s)f'(s)+S(s)g'(s)]\}-[2\Omega'^2-\Omega'(s)\Omega'-\Omega'^2(s)]$$

$$\{S[C(s)f'(s)+S(s)g'(s)]-C[C(s)f'(s)+S(s)g'(s)]\}\Big\}\frac{\mu(1-\gamma)}{\tau_R}\exp\left(-\frac{t-s}{\tau_R}\right).$$

Let the interval $[0, t]$ be partitioned into n subintervals $[t_1 = 0, t_2, \ldots, t_n = t]$. The third integral in eqn (22) can be written as

$$\int_0^t G_4[\Omega,\Omega',\Omega'';\Omega(s),\Omega'(s),\Omega''(s);f';f'(s);g'(s);t-s]\,ds$$

$$= \int_{t_1}^{t_n} G_4[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);\Omega(s),\Omega'(s),\Omega''(s);f'(t_n);f'(s);g'(s);t_n-s]\,ds. \quad (24)$$

Expressing (24) as a summation of $n-1$ integrals over the subintervals (t_k, t_{k+1}), $(1 \leqslant k \leqslant n-1)$, approximating each of these by the trapezoidal rule and denoting the finite sum approximation to the third integral of eqn (22) by Σ_4, one finds

$$\Sigma_4 = \int_0^t G_4[\Omega,\Omega',\Omega'';\Omega(s),\Omega'(s),\Omega''(s);f';f'(s);g'(s);t-s]\,ds$$

$$\approx \tfrac{1}{2}\Big\{G_4[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);\Omega(t_n),\Omega'(t_n),\Omega''(t_n);f'(t_n);f'(t_n);$$

$$g'(t_n);0](t_n-t_{n-1})+G_4[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);\Omega(t_1),\Omega'(t_1),\Omega''(t_1);$$

$$f'(t_n);f'(t_1);g'(t_1);t_n-t_1](t_2-t_1)+\sum_{k=2}^{n-1}G_4[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);$$

$$\Omega(t_k),\Omega'(t_k),\Omega''(t_k);f'(t_n);f'(t_k);g'(t_k);t_n-t_k](t_{k+1}-t_{k-1})\Big\}. \quad (25)$$

Similarly, the rest of the integrals in eqns (21)–(23) can also be written as expression (25). For notational convenience, the finite sum approximations to the integrals are denoted by Σ_i. Letting Σ_i^j denote only the terms in the approximations which contain t_k, $k < n$, it follows that

$$\Sigma_5 = \int_0^t G_5[\Omega,\Omega',\Omega'';\Omega(s),\Omega'(s),\Omega''(s);f'(s);g';g'(s);t-s]\,\mathrm{d}s$$

$$\approx \tfrac{1}{2}\Big\{G_5[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);\Omega(t_n),\Omega'(t_n),\Omega''(t_n);f'(t_n);g'(t_n);$$

$$g'(t_n);0](t_n-t_{n-1})+G_5[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);\Omega(t_1),\Omega'(t_1),\Omega''(t_1);$$

$$f'(t_1);g'(t_n);g'(t_1);t_n-t_1](t_2-t_1)+\sum_{k=2}^{n-1}G_5[\Omega(t_n),\Omega'(t_n),\Omega''(t_n);$$

$$\Omega(t_k),\Omega'(t_k),\Omega''(t_k);f'(t_k);g'(t_n);g'(t_k);t_n-t_k](t_{k+1}-t_{k-1})\Big\}, \tag{26}$$

and

$$\int_0^t \frac{\partial\Omega'(s)}{\partial Z}G_1[t-s]\,\mathrm{d}s \approx \frac{\partial\Omega'(t_n)}{\partial Z}\bar{G}_1[0]+\Sigma_1^{\Omega}, \tag{27}$$

$$\int_0^t \frac{\partial f''(s)}{\partial Z}G_2[\Omega;\Omega(s);t-s]\,\mathrm{d}s \approx \frac{\partial f''(t_n)}{\partial Z}\bar{G}_2[\Omega(t_n);\Omega(t_n);0]+\Sigma_2^f, \tag{28}$$

$$\int_0^t \frac{\partial f''(s)}{\partial Z}G_3[\Omega;\Omega(s);t-s]\,\mathrm{d}s \approx \frac{\partial f''(t_n)}{\partial Z}\bar{G}_3[\Omega(t_n);\Omega(t_n);0]+\Sigma_3^f, \tag{29}$$

$$\int_0^t \frac{\partial g''(s)}{\partial Z}G_2[\Omega;\Omega(s);t-s]\,\mathrm{d}s \approx \frac{\partial g''(t_n)}{\partial Z}\bar{G}_2[\Omega(t_n);\Omega(t_n);0]+\Sigma_2^g, \tag{30}$$

$$\int_0^t \frac{\partial g''(s)}{\partial Z}G_3[\Omega;\Omega(s);t-s]\,\mathrm{d}s \approx \frac{\partial g''(t_n)}{\partial Z}\bar{G}_3[\Omega(t_n);\Omega(t_n);0]+\Sigma_3^g, \tag{31}$$

where

$$\bar{G}_1[0] = \tfrac{1}{2}G_1[0](t_n-t_{n-1}),$$

$$\bar{G}_2[\Omega(t_n);\Omega(t_n);0] = \tfrac{1}{2}G_2[\Omega(t_n);\Omega(t_n);0](t_n-t_{n-1}),$$

$$\bar{G}_3[\Omega(t_n);\Omega(t_n);0] = \tfrac{1}{2}G_3[\Omega(t_n);\Omega(t_n);0](t_n-t_{n-1}),$$

$$\Sigma_1^{\Omega} = \frac{1}{2}\left\{G_1[t_n-t_1](t_2-t_1)\frac{\partial\Omega'(t_1)}{\partial Z}+\sum_{k=2}^{n-1}G_1[t_n-t_k](t_{k+1}-t_{k-1})\frac{\partial\Omega'(t_k)}{\partial Z}\right\},$$

$$\Sigma_2^f = \frac{1}{2}\Big\{G_2[\Omega(t_n);\Omega(t_1);t_n-t_1](t_2-t_1)\frac{\partial f''(t_1)}{\partial Z}$$

$$+\sum_{k=2}^{n-1} G_2[\Omega(t_n);\Omega(t_k);t_n-t_k](t_{k+1}-t_{k-1})\frac{\partial f''(t_k)}{\partial Z}\Bigg\},$$

$$\Sigma_3^f=\frac{1}{2}\Bigg\{G_3[\Omega(t_n);\Omega(t_1);t_n-t_1](t_2-t_1)\frac{\partial f''(t_1)}{\partial Z}$$

$$+\sum_{k=2}^{n-1} G_3[\Omega(t_n);\Omega(t_k);t_n-t_k](t_{k+1}-t_{k-1})\frac{\partial f''(t_k)}{\partial Z}\Bigg\},$$

$$\Sigma_2^g=\frac{1}{2}\Bigg\{G_2[\Omega(t_n);\Omega(t_1);t_n-t_1](t_2-t_1)\frac{\partial g''(t_1)}{\partial Z}$$

$$+\sum_{k=2}^{n-1} G_2[\Omega(t_n);\Omega(t_k);t_n-t_k](t_{k+1}-t_{k-1})\frac{\partial g''(t_k)}{\partial Z}\Bigg\},$$

$$\Sigma_3^g=\frac{1}{2}\Bigg\{G_3[\Omega(t_n);\Omega(t_1);t_n-t_1](t_2-t_1)\frac{\partial g''(t_1)}{\partial Z}$$

$$+\sum_{k=2}^{n-1} G_3[\Omega(t_n);\Omega(t_k);t_n-t_k](t_{k+1}-t_{k-1})\frac{\partial g''(t_k)}{\partial Z}\Bigg\}.$$

It can be seen, from the last five equations, that the corresponding terms of eqns (21)–(23) depending on $\Omega(Z, t_k)$ have

$$\frac{\partial f''(Z,t_k)}{\partial Z} \quad \text{or} \quad \frac{\partial g''(Z,t_k)}{\partial Z},$$

as a coefficient. For $t_k < t_n$, this derivative is approximated by a simple forward difference expression. From eqns (21)–(23), the initial elastic response, $t_n = t_1 = 0$, satisfies the following system of equations:

$$\frac{\partial \Omega(t_1)}{\partial Z}=\Omega_1(t_1),$$

$$\frac{\partial \Omega_1(t_1)}{\partial Z}=\Omega_2(t_1)=0,$$

$$\frac{\partial f(t_1)}{\partial Z}=f_1(t_1),$$

$$\frac{\partial f_1(t_1)}{\partial Z}=f_2(t_1),$$

$$\frac{\partial f_2(t_1)}{\partial Z}=-\frac{F_2[\Omega_1(t_1),\Omega_2(t_1);f_1(t_1);g_1(t_1);g_2(t_1);t_1]}{F_1[t_1]}, \tag{32}$$

$$\frac{\partial g(t_1)}{\partial Z}=g_1(t_1),$$

$$\frac{\partial g_1(t_1)}{\partial Z}=g_2(t_1),$$

$$\frac{\partial g_2(t_1)}{\partial z}=-\frac{F_3[\Omega_1(t_1),\Omega_2(t_1);f_1(t_1),f_2(t_1);g_1(t_1);t_1]}{F_1[t_1]}.$$

This system can be integrated by a fourth order Runge–Kutta method [cf. William (1986)] subject to the conditions given by (17)–(20).

The specific values of $\Omega'(-H, t_1)$, $f'(-H, t_1)$ $(=f_1(-H, t_1))$, $f''(-H, t_1)$ $(=f_2(-H, t_1))$, $g'(-H, t_1)$ $(=g_1(-H, t_1))$ and $g''(-H, t_1)$ $(=g_2(-H, t_1))$ are determined so that $\Omega(H, t_1)$, $f(H, t_1)$, and $g(H, t_1)$ have the prescribed values, and the two conditions in eqn (20) are satisfied. The starting value of $\Omega(-H, t_1)$ for the numerical integration is given by the first equation of (17) and $\Omega'(-H, t_1)$ is adjusted so as to satisfy the second equation of (17). For the displacement functions, f and g, the starting values of $f(-H, t_1)$ and $g(-H, t_1)$ are given by the first equations of (18) and (19), respectively. The starting values of $f'(-H, t_1)$, $f''(-H, t_1)$, $g'(-H, t_1)$ and $g''(-H, t_1)$ are determined so as to satisfy the second equation of (18) and (19) and the conditions of (20), which requires a four-dimensional shooting. The four-dimensional shooting can be implemented by two coupled two-dimensional shooting methods for f and g, respectively. First, two values are assumed for $g'(-H, t_1)$ and $g''(-H, t_1)$. While the two values are fixed, $f'(-H, t_1)$ and $f''(-H, t_1)$ are adjusted so as to satisfy the second equation of (18) and the first condition of (20). With the updated values of $f'(-H, t_1)$ and $f''(-H, t_1)$ which are fixed, $g'(-H, t_1)$ and $g''(-H, t_1)$ are adjusted so as to satisfy the second equation of (19) and the second condition of (20). The process is repeated with updated values of $g'(-H, t_1)$ and $g''(-H, t_1)$ as new fixed values until the second equations of (18) and (19) and the two conditions of (20) are met simultaneously. For the details of the two-dimensional shooting of numerical simulation, the reader is referred to the recent paper by Dai and Rajagopal (1993).

For $n > 2$, eqns (21)–(23) are written as

$$\frac{\partial \Omega'(t_n)}{\partial Z}\{F_1[t_n]+\bar{G}_1[0]\}+\Sigma_1^{\Omega} = 0, \tag{33}$$

$$\frac{\partial f''(t_n)}{\partial Z}\{F_1[t_n]+\bar{G}_2[\Omega(t_n);\Omega(t_n);0]\}+\frac{\partial g''(t_n)}{\partial Z}\bar{G}_3[\Omega(t_n);\Omega(t_n);0]$$
$$+F_2[\Omega'(t_n),\Omega''(t_n);f'(t_n);g'(t_n),g''(t_n);t_n]+\Sigma_2^f+\Sigma_3^g+\Sigma_4 = 0, \tag{34}$$

$$\frac{\partial g''(t_n)}{\partial Z}\{F_1[t_n]+\bar{G}_2[\Omega(t_n);\Omega(t_n);0]\}+\frac{\partial f''(t_n)}{\partial Z}\bar{G}_3[\Omega(t_n);\Omega(t_n);0]$$
$$+F_3[\Omega'(t_n),\Omega''(t_n);f'(t_n),f''(t_n);g'(t_n);t_n]+\Sigma_2^g+\Sigma_3^f+\Sigma_5 = 0. \tag{35}$$

From eqns (34) and (35), one finds that

$$\frac{\partial f''(t_n)}{\partial Z}[(F_1+\bar{G}_2)^2-\bar{G}_3^2]+(\Sigma_2^f+\Sigma_3^g+\Sigma_4+F_2)(F_1+\bar{G}_2)-(\Sigma_2^g+\Sigma_3^f+\Sigma_5+F_3)\bar{G}_3 = 0, \tag{36}$$

$$\frac{\partial g''(t_n)}{\partial Z}[(F_1+\bar{G}_2)^2-\bar{G}_3^2]+(\Sigma_2^g+\Sigma_3^f+\Sigma_5+F_3)(F_1+\bar{G}_2)-(\Sigma_2^f+\Sigma_3^g+\Sigma_4+F_2)\bar{G}_3 = 0, \tag{37}$$

where

$$F_1 = F_1[t_n],\quad F_2 = F_2[\Omega'(t_n),\Omega''(t_n);f'(t_n);g'(t_n),g''(t_n);t_n],$$
$$F_3 = F_3[\Omega'(t_n),\Omega''(t_n);f'(t_n),f''(t_n);g'(t_n);t_n],$$
$$\bar{G}_1 = \bar{G}_1[0],\quad \bar{G}_2 = \bar{G}_2[\Omega(t_n);\Omega(t_n);0],\quad \bar{G}_3 = \bar{G}_3[\Omega(t_n);\Omega(t_n);0].$$

Finally, eqns (33), (36) and (37) are written in the form:

$$\frac{\partial\Omega(t_n)}{\partial Z} = \Omega_1(t_n),$$

$$\frac{\partial\Omega_1(t_n)}{\partial Z} = -\frac{\Sigma_1^{\Omega}}{F_1+\bar{G}_1},$$

$$\frac{\partial f(t_n)}{\partial Z} = f_1(t_n),$$

$$\frac{\partial f_1(t_n)}{\partial Z} = f_2(t_n),$$

$$\frac{\partial f_2(t_n)}{\partial Z} = \frac{(\Sigma_2^g+\Sigma_3^f+\Sigma_5+F_3)\bar{G}_3-(\Sigma_2^f+\Sigma_3^g+\Sigma_4+F_2)(F_1+\bar{G}_2)}{(F_1+\bar{G}_2)^2-G_3^2}, \tag{38}$$

$$\frac{\partial g(t_n)}{\partial Z} = g_1(t_n),$$

$$\frac{\partial g_1(t_n)}{\partial Z} = g_2(t_n),$$

$$\frac{\partial g_2(t_n)}{\partial Z} = \frac{(\Sigma_2^f+\Sigma_3^g+\Sigma_4+F_2)\bar{G}_3-(\Sigma_2^g+\Sigma_3^f+\Sigma_5+F_3)(F_1+\bar{G}_2)}{(F_1+\bar{G}_2)^2-\bar{G}_3^2}.$$

In the above system, F_1, F_2, F_3, $\bar{G}_1$, $\bar{G}_2$ and $\bar{G}_3$ depend only on $\Omega(t_n)$, $\Omega'(t_n)$, $\Omega''(t_n)$, $f'(t_n)$, $f''(t_n)$, $g'(t_n)$ and $g''(t_n)$, while Σ_2^f, Σ_3^f, Σ_2^g, Σ_3^g, Σ_4 and Σ_5 depend on $\Omega(t_n)$, $\Omega'(t_n)$, $\Omega''(t_n)$, $f'(t_n)$, $g'(t_n)$, $\Omega(t_k)$, $\Omega'(t_k)$, $\Omega''(t_k)$, $f'(t_k)$, $g'(t_k)$,

$$\frac{\partial f''(t_k)}{\partial Z} \quad \text{and} \quad \frac{\partial g''(t_k)}{\partial Z}, k < n.$$

Because the last seven have been found by solving the system (38) for times $t_k < t_n$, Σ_2^f, Σ_3^f, Σ_2^g, Σ_3^g, Σ_4 and Σ_5 may now be considered as functions of the independent variables Z, $\Omega(t_n)$, $\Omega'(t_n)$, $\Omega''(t_n)$, $f'(t_n)$, $g'(t_n)$. Thus, for each time t_n, $\Omega(t_n)$, $\Omega'(t_n)$, $\Omega''(t_n)$, $f(t_n)$, $f'(t_n)$, $f''(t_n)$, $g(t_n)$, $g'(t_n)$, $g''(t_n)$,

$$\frac{\partial f''(t_n)}{\partial Z} \quad \text{and} \quad \frac{\partial g''(t_n)}{\partial Z}$$

are found by solving the coupled system of nonlinear ordinary differential equations, (38), and stored for the next step. The solution of the system (38), subject to the boundary conditions at $Z = -H$ and $Z = H$, is obtained by the same procedure as was outlined for the initial elastic response. Once the functions $\Omega(Z, t)$, $f(Z, t)$ and $g(Z, t)$ are determined, $x(Z, t)$ and $y(Z, t)$ are obtained directly from eqn (1), while the stresses are calculated from eqn (8).

5. DISCUSSION

Numerical results are presented in this section. Simulations conducted in this work correspond to the following non-dimensional quantities:

$$\bar{z} = \frac{z}{2H}, \quad \bar{Z} = \frac{Z}{2H}, \quad \bar{\Omega}' = 2\Omega' H, \quad \bar{\Omega}'' = 4\Omega'' H^2, \quad \bar{f} = \frac{f}{2H}, \quad \bar{f}'' = 2f'' H, \quad \bar{f}''' = 4f''' H^2,$$

$$\bar{g} = \frac{g}{2H}, \quad \bar{g}'' = 2g''H, \quad \bar{g}''' = 4g'''H^2, \quad \bar{\sigma}_{xz} = \frac{\sigma_{xz}}{C_0}, \quad \bar{\sigma}_{yz} = \frac{\sigma_{yz}}{C_0}, \quad \bar{M} = \frac{M}{2HC_0}, \quad \bar{r} = \frac{r}{2H},$$

where M is the local resultant moment which will be defined later. The parameter μ is chosen to be 0.1, the ratio of long time to initial modulus γ is chosen to be 0.25, and the relaxation time τ_R is assumed to be 1.0. For the problem under consideration, it is assumed that $\Omega_0(t) = (\pi/16)$ and $a(t) = 0.2H$. At each time step, the value of $\bar{\Omega}'(-0.5, t)$ is accepted as the final value when $|\bar{\Omega}(0.5, t) - \pi/16.0| < \varepsilon_1$. Similarly, the values of $\bar{f}'(-0.5, t)$, $\bar{f}''(-0.5, t)$, $\bar{g}'(-0.5, t)$ and $\bar{g}''(-0.5, t)$ are accepted as the final values when $|\bar{f}(0.5, t) - 0.1| < \varepsilon_2$, $|\bar{g}(0.5, t)| < \varepsilon_3$, $|\bar{f}'(-0.5, t) - \bar{f}'(0.5, t)| < \varepsilon_4$ and $|\bar{g}'(-0.5, t) - \bar{g}'(0.5, t)| < \varepsilon_5$, where $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 1.0 \times 10^{-4}$. Time steps t_k are chosen to vary logarithmically [cf. Wineman (1972) and Dai and Rajagopal (1993)]. This permits small time increments during early times when quantities are undergoing larger variation and larger time increments for later times when the variations are smaller. The time steps used are given by the relation $t_{k+1} = t_k \times 10^A$, where $A = 0.2$ for $k = 2, 3, \ldots, 10$, $A = 0.05$ for $k > 11$, with $t_2 = 0.01$.

Figure 3 shows the variation of Ω across the slab thickness and with time t, for $t = 0$, $t = 0.5$, $t = 1.0$ and $t \to \infty$. It is interesting to note that Ω changes with $\bar{Z}$ linearly all the time. It means that $\bar{\Omega}'(\bar{Z}, t)$ is a constant across the slab thickness which implies that the planes parallel to the boundary plates with the same distance of separation to each other rotate by a constant relative angle. Figure 4 depicts the variation of $\bar{f}$ versus $\bar{Z}$ and t. It shows that $\bar{f}$ is symmetric about the origin of the coordinate system x, y and z. The absolute values of $\bar{f}$ decrease as time increases, except at the point in the midplane ($\bar{Z} = 0$) and the points in the boundary surfaces ($\bar{Z} = \pm 0.5$) where $\bar{f}$ has the fixed values 0.0, 0.1 and -0.1, respectively. The displacements of the material points at the centers of rotations decrease in the X-direction as time elapses, due to relaxation. Figure 5 shows the variation of $\bar{g}$ with $\bar{Z}$ and t. Opposite to $\bar{f}$ the absolute values of $\bar{g}$ increase as time elapses, without displacements at the three fixed points. The material points on the locus of the rotation centers move toward certain positions in the Y-direction from their original positions (before deformation). From Figs 4 and 5, it is concluded that the locus of the centers of rotation, $\bar{f}$ and $\bar{g}$, is a curve in space which is symmetric about the origin of the coordinate system all the time. The position of the curve in space, $\sqrt{\bar{f}^2 + \bar{g}^2}$, changes with time and tends to a certain position as time elapses. Figure 6 depicts the path histories of the particles originally at $X = Y = 0$, where $t = 0$ corresponds to elastic response. It is seen that only the particle at $X = Y = Z = 0$ keeps its original position.

The local resultant moment on the upper boundary of the circle $\sqrt{X^2 + Y^2} = r$ about the rotating center $\bar{f}(0.5, t)$ is defined as (cf. Fig. 7)

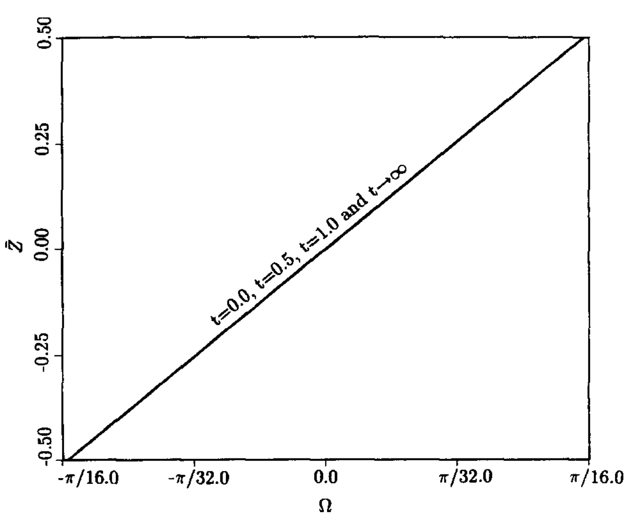

Fig. 3. Variation of Ω with $\bar{Z}$ and t.

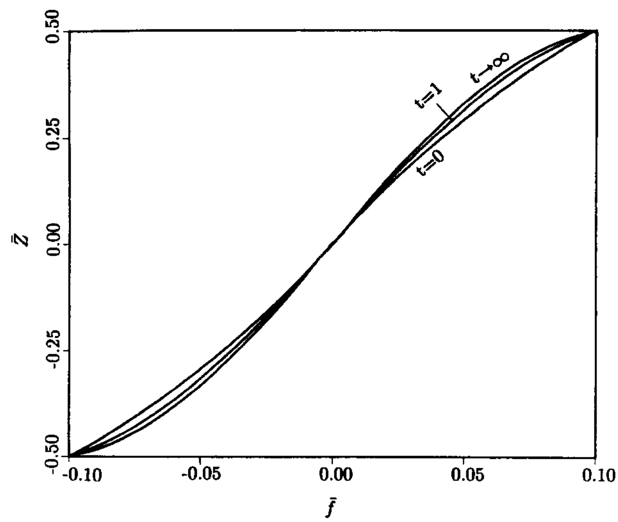

Fig. 4. Variation of $\bar{f}$ with $\bar{Z}$ and t.

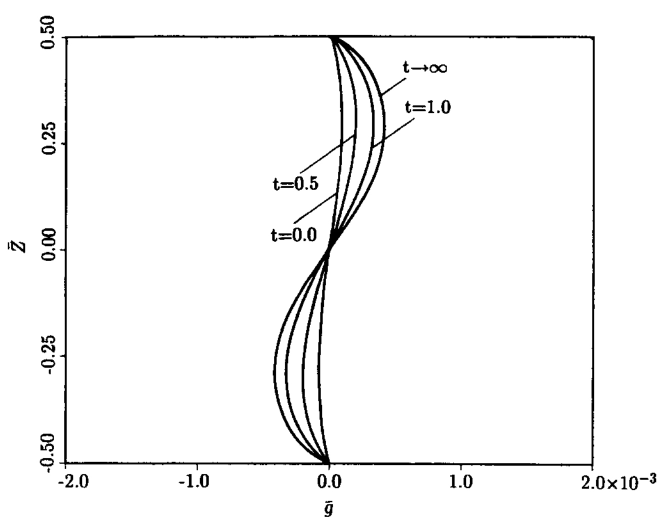

Fig. 5. Variation of $\bar{g}$ with $\bar{Z}$ and t.

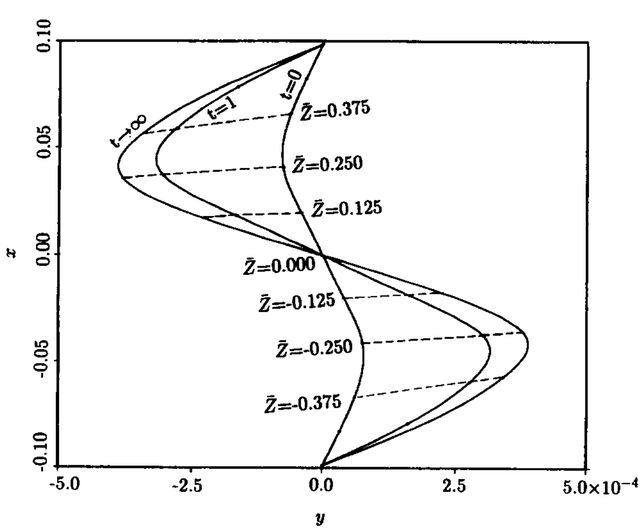

Fig. 6. Particle paths.

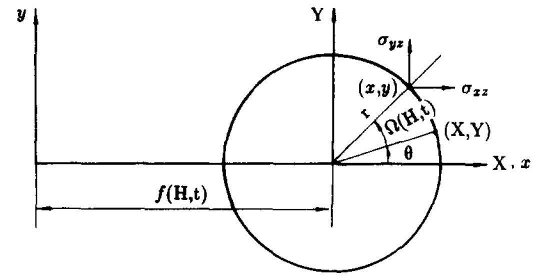

Fig. 7. Local resultant moment on the top boundary.

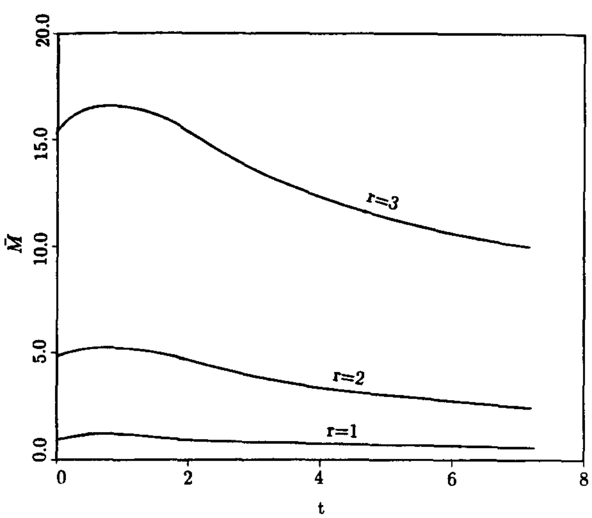

Fig. 8. Variation of local resultant moment with time.

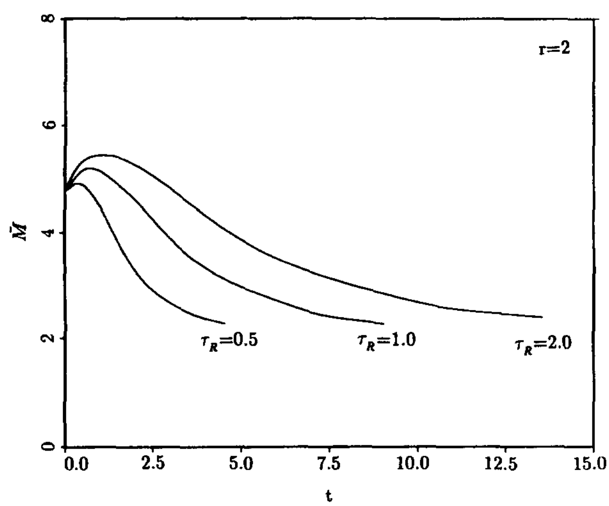

Fig. 9. Effect of relaxation time on resultant moment.

$$M = r\int_0^{2\pi} [\sigma_{yz}\cos(\Omega_0+\theta) - \sigma_{xz}\sin(\Omega_0+\theta)]\,d\theta.$$

Figure 8 shows the variation of the dimensionless form of the above defined local resultant moment, $\bar{M}$, with time for $\bar{r} = 1$, 2 and 3. It is seen that the local resultant moments increase from the values of elastic response, reach maximum and then decrease asymptotically to certain values as time elapses, due to the material relaxation of the viscoelastic slab. The effect of the relaxation time on resultant moments is shown in Fig. 9.

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